REPRESENTATION OF SET VALUED OPERATORS

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ABSTRACT. In this paper we prove representation theorems for set valued additive operators acting on the spaces L_X^1 (X = separable Banach space), L^1 and L^∞. Those results generalize well-known ones for single valued operators and among them the celebrated Dunford-Pettis theorem. The properties of these representing integrals are studied. We also have a differentiability result for multifunctions analogous to the one that says that an absolutely continuous function from a closed interval into a Banach space with the Radon-Nikodým property is almost everywhere differentiable and also it is the primitive of its strong derivative. Finally we have a necessary and sufficient condition for the set of integrable selectors of a multifunction to be w-compact in L_X^1. This result is a new very general result about w-compactness in the Lebesgue-Bochner space L_X^1.

1. Introduction. Integral representation theory has been studied by several authors for linear and nonlinear additive functionals and operators on various function spaces such as Lebesgue-Bochner spaces and Orlicz spaces (see Alò and Korvin [1], Diestel and Uhl [12], Hiai [17] and Mizel and Sundaresan [22]).

In this paper we examine the corresponding problems for additive set valued operator, from which we obtain an integral representation theorem for a lower semicontinuous set valued operator acting on the Lebesgue-Bochner space $L_X^1(\Omega)$. In fact we show that the representing integrand multifunction has several nice regularity properties. We also extend the celebrated Dunford-Pettis theorem which is a result about the integral representation of a linear continuous operator on L^1 into a Banach space X (see Diestel and Uhl [12, p. 73]) to the case of an additive set valued operator (also called multioperator). Several versions of this result are presented under a different and very general set of hypotheses.

In §4 we examine some additional properties of such representable set valued operators. Also we present a general result about "differentiation" of multifunctions. It is well known that if X is a Banach space with the Radon-Nikodým Property (abbreviated by R.N.P.), then a function $f: [0, T] \to X$ which is absolutely continuous is almost everywhere strongly differentiable, $\dot{f}(\cdot)$ is Bochner integrable and furthermore we can write that $f(t) = f(0) + \int_0^t \dot{f}(\tau)\, d\tau$. This result was first proved by Gelfand [21] for the case X is reflexive and later it was realized that the result is

still true under the more general hypothesis that X has the R.N.P. In fact it turns out that this is also a sufficient condition for a space to be a Radon-Nikodým space. A multivalued version of this result was obtained by Artstein [2] for $X = \mathbf{R}^n$ and by Castaing [5] for X being a separable dual Banach space and under quite restrictive additional hypotheses. Here we relax all those assumptions and we present a more general differentiability result. Finally we also have an interesting result about the set of integrable selectors of a measurable multifunction.

In the next section we establish our terminology and notations and we present some auxiliary material that we will need in the sequel.

2. Preliminaries. Let (Ω, Σ, μ) be a complete, finite measure space and X a separable Banach space. Additional hypotheses will be introduced as needed. By X^* we will denote the topological dual of X. We will use the following notation.

$P_{f(c)}(X) = \{ A \subseteq X \colon \text{nonempty, closed, (convex)} \}$.

$P_{k(c)}(X) = \{ A \subseteq X \colon \text{nonempty, compact, (convex)} \}$.

A w in front of f (resp. k) will mean that the closedness (resp. compactness) is with respect to the weak topology $w(X, X^*)$. If $A \in P_f(X)$, then we write $|A| = \sup_{x \in A} \|x\|$. Also $\sigma_A(x^*) = \sup_{x \in A}(x^*, x)$ is the support function of A.

A multifunction $F \colon \Omega \to P_f(X)$ is said to be measurable if it satisfies any of the following three equivalent conditions:

(1) $F^-(U) = \{ \omega \in \Omega \colon F(\omega) \cap U \neq \varnothing \} \in \Sigma$ for all $U \subseteq X$ open.

(2) $\omega \to d(x, F(\omega))$ is Σ-measurable for all $x \in X$.

(3) There exists a sequence $\{ f_n(\cdot) \}_{n \geqslant 1}$ of measurable functions $f_n \colon \Omega \to X$ s.t. $f_n(\omega) \in F(\omega)$ and $F(\omega) = \text{cl}\{ f_n(\omega) \}_{n \geqslant 1}$ for all $\omega \in \Omega$ (Castaing representation).

A measurable function $f \colon \Omega \to X$ s.t. $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ is said to be a selector of $F(\cdot)$. By S_F^1 we denote all selectors of $F(\cdot)$ belonging to $L_X^1(\Omega)$, i.e., $S_F^1 = \{ f(\cdot) \in L_X^1(\Omega) \colon f(\omega) \in F(\omega) \ \mu\text{-a.e.} \}$. We will say that $F(\cdot)$ is integrably bounded if and only if $|F(\cdot)| \in L^1(\Omega)$. In this case the Kuratowski-Ryll-Nardzewski selection theorem tells us that $S_F^1 \neq \varnothing$. Using S_F^1 we can define an integral for multifunctions. This was first introduced by Aumann [3] for the case $X = \mathbf{R}^n$ and is the natural extension of the single valued integral and of the Minkowski sum of sets. So we have

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{ \int_\Omega f(\omega)\, d\mu(\omega) \colon f \in S_F^1 \right\},$$

where the vector integrals are taken in the sense of Bochner. This integral turned out to be the appropriate analytical tool in several applied fields like optimization, optimal control, mathematical economics and statistics. For more details about measurable multifunctions and their integrals we refer to Castaing and Valadier [6], Hiai and Umegaki [18], Himmelberg [19] and Rockafellar [25].

Finally we would like to introduce a notion of convergence of nonempty, closed sets in X which we will call Kuratowski-Mosco convergence (K-M convergence) and which is more general then convergence in the Hausdorff metric. Let $\{ A_n \}_{n \geqslant 1}$ be a sequence in $P_f(X)$. Let

$$\text{s-}\liminf_{n \to \infty} A_n = \left\{ x \in X \colon x = \text{s-}\lim_{n \to \infty} x_n, \ x_n \in A_n, n \geqslant 1 \right\}$$

(here s denotes the strong (norm) topology on X) and

$$\text{w-}\limsup_{n \to \infty} A_n = \left\{ x \in X : x = \text{w-}\lim_{k \to \infty} x_{n_k}, \, x_{n_k} \in A_{n_k}, k \geqslant 1 \right\}.$$

We will say that A_n converges to $A \in P_f(X)$ in the Kuratowski-Mosco sense (written as $A_n \xrightarrow{\text{K-M}} A$ as $n \to \infty$) if and only if $\text{w-}\limsup_{n \to \infty} A_n = A = \text{s-}\liminf_{n \to \infty} A_n$. For more details we refer to Mosco [23].

3. Representation theorems. Consider a set valued operator $T: L^1_X(\Omega) \to 2^X \setminus \{ \varnothing \}$. We will say that $T(\cdot)$ is additive if $T(f_1 + f_2) = T(f_1) + T(f_2)$ for all $f_1(\cdot)$, $f_2(\cdot) \in L^1_X(\Omega)$ s.t. $\text{supp}(f_1) \cap \text{supp}(f_2) = \varnothing$ (i.e. f_1, f_2 have disjoint supports). We will say that $T(\cdot)$ is sublinear if it is additive in the above sense and positively homogeneous.

We start with a lemma that is also interesting in its own sake.

LEMMA. *If $F: \Omega \times X \to P_{wkc}(X)$ is a multifunction s.t.*
(1) *$F(\cdot, x)$ is measurable for all $x \in X$,*
(2) *$x \to \sigma_{F(\omega, x)}(x^*)$ is l.s.c. for all $\omega \in \Omega$ and all $x^* \in X^*$, $\sigma_{F(\cdot, \cdot)}(x^*)$ separable*
then for all $f: \Omega \to X$ measurable, $\omega \to F(\omega, f(\omega))$ is measurable.

PROOF. Consider the function $\varphi_{x^*}(\omega, x) = \sigma_{F(\omega, x)}(x^*)$ and fix $x^* \in X^*$. We claim that $\varphi_{x^*}(\cdot, \cdot)$ is a normal integrand. From Rockafellar [25] we know that it suffices to show that the multifunction $\omega \to R(\omega) = \text{epi } \varphi_{x^*}(\omega, \cdot)$ is closed valued and measurable. From hypothesis (2) we have that $x \to \varphi_{x^*}(\omega, x)$ is l.s.c. and so $R(\cdot)$ is closed valued. Next note that $\varphi_{x^*}(\omega, \cdot)$ is finite valued for all $\omega \in \Omega$. So $\text{dom } R = \Omega$. Let $D = \{(x_n, \lambda_n) \in X \times \mathbf{R}; n \geqslant 1\}$ be a countable dense set in $X \times \mathbf{R}$ and define the functions $g_n(\omega) = (x_n, \max(\varphi_{x^*}(\omega, x_n), \lambda_n))$. Clearly $g_n(\cdot)$ is a Castaing representation for $R(\cdot)$. So $\omega \to R(\omega)$ is measurable. Thus we deduce that $\varphi_{x^*}(\cdot, \cdot)$ is a normal integrand which means that it is $(\Sigma \times B(X))$-measurable. So $\omega \to \varphi_{x^*}(\omega, f(\omega)) = \sigma_{F(\omega, f(\omega))}(x^*)$ is measurable and this means, by Theorem III-37 of Castaing and Valadier [6], that $\omega \to F(\omega, f(\omega))$ is measurable.   Q.E.D.

REMARK. In the above proof we also got that $(\omega, x) \to F(\omega, x)$ is $(\Sigma \times B(X))$-measurable. Also if $F(\omega, \cdot)$ is l.s.c. (in the set valued sense, see Castaing and Valadier [6]), then $x \to \sigma_{F(\omega, x)}(x^*)$ is l.s.c.

Now we are ready for the first representation theorem.

THEOREM 3.1. *Assume that X is reflexive. If $T: L^1_X(\Omega) \to P_{fc}(X)$ is sublinear and l.s.c., and $T(f) \subseteq \int_\Omega \|f(\omega)\| W(\omega) \, d\mu(\omega)$ for all $f(\cdot) \in L^1_X(\Omega)$, where $W: \Omega \to P_{wkc}(X)$ is integrably bounded, then there exists $F: \Omega \to P_{wkc}(X)$ s.t.*
(a) *for all $x \in X$, $F(\cdot, \cdot)$ is integrably bounded by $p(\cdot) \in L^1(\Omega)$,*
(b) *for all $\omega \in \Omega$, $F(\omega, \cdot)$ is l.s.c. from X into (X, w) and $F(\omega, 0) = \{0\}$ μ-a.e.,*
(c) *for all $f(\cdot) \in L^1_X(\Omega)$ we have that $F(\omega, f(\omega)) \subseteq \|f(\omega)\| W(\omega)$ μ-a.e. and $T(f) = \int_\Omega F(\omega, f(\omega)) \, d\mu(\omega)$.*

PROOF. In the light of the previous lemma and of the Kuratowski-Ryll-Nardzewski selection theorem, the integral in the conclusion of the theorem is nonempty.

For every $x^* \in X^*$ consider the function $f \to \sigma_{T(f)}(x^*)$. Clearly this is sublinear since $T(\cdot)$ is. We claim that it is also lower semicontinuous (l.s.c.). To show this we work as follows. Let $f_n \xrightarrow{L^1_X} f$ and let $x \in$ s-$\underline{\lim}_{n \to \infty} T(f_n)$. Then we can find $x_n \in T(f_n)$ s.t. $x_n \xrightarrow{s} x \Rightarrow (x^*, x_n) \to (x^*, x)$. But $(x^*, x_n) \leqslant \sigma_{T(f_n)}(x^*)$ for all $n \geqslant 1 \Rightarrow (x^*, x) \leqslant \underline{\lim}_{n \to \infty} \sigma_{T(f_n)}(x^*)$. Since $x \in$ s-$\underline{\lim}_{n \to \infty} T(f_n)$ was arbitrary we conclude that $\sigma_{\text{s-}\underline{\lim}_{n \to \infty} T(f_n)}(x^*) \leqslant \underline{\lim}_{n \to \infty} \sigma_{T(f_n)}(x^*)$. On the other hand, since by hypothesis $T(\cdot)$ is l.s.c. we have that

$$T(f) \subseteq \text{s-}\underline{\lim_{n \to \infty}} T(f_n) \Rightarrow \sigma_{T(f)}(x^*) \leqslant \sigma_{\text{s-}\underline{\lim}_{n \to \infty} T(f_n)}(x^*).$$

So indeed $f \to \sigma_{T(f)}(x^*)$ is l.s.c. Then Theorem 5.1 of Hiai [17] tells us that for all $x^* \in X^*$ there exists $\varphi_{x^*} \colon \Omega \times \Sigma \to \overline{\mathbb{R}}$ normal s.t. $\varphi_{x^*}(\omega, 0) = 0$ μ-a.e. and $\sigma_{T(f)}(x^*) = \int_\Omega \varphi_{x^*}(\omega, x)\, d\mu(\omega)$. Fix $x^* \in X^*$ and $x \in X$. Then by letting $f = \chi_A x$, where $A \in \Sigma$, we have that

$$\sigma_{T(\chi_A x)}(x^*) = \int_A \varphi_{x^*}(\omega, x)\, d\mu(\omega) \leqslant \|x\| \|x^*\| \int_A |W(\omega)|\, d\mu(\omega).$$

Since this is true for all $A \in \Sigma$ we have that $\varphi_{x^*}(\omega, x) \leqslant \|x\| \|x^*\| |W(\omega)|$ for all $\omega \in \Omega \setminus N_{xx^*}$ with $\mu(N_{xx^*}) = 0$. Set $\Omega_n = \{\omega \in \Omega \colon n \leqslant |W(\omega)| < n + 1\}$; $\varphi_{x^*}(\cdot, x)$ is essentially bounded on every Ω_n. Let ρ be a positive, linear lift of $L^\infty(\Omega_n, \Sigma_n, \mu|_{\Omega_n})$ (where $\Sigma_n = \Sigma \cap \Omega_n$). Such a lift exists by Ionescu-Tulcea [20]. Let $\hat{\varphi}_{x^*}(\omega, x) = \rho[\varphi_{x^*}(\omega, x)]$. Define a function on all Ω by piecing those lifts together. Then since

$$\int_A \varphi_{x_1^* + x_2^*}(\omega, x)\, d\mu(\omega) \leqslant \int_A \left[\varphi_{x_1^*}(\omega, x) + \varphi_{x_2^*}(\omega, x) \right] d\mu(\omega)$$

and

$$\int_A \varphi_{\lambda x^*}(\omega, x)\, d\mu(\omega) = \lambda \int_A \varphi_{x^*}(\omega, x)\, d\mu(\omega)$$

for all $A \in \Sigma$, we get that $\hat{\varphi}_{x_1^* + x_2^*}(\omega, x) \leqslant \hat{\varphi}_{x_1^*}(\omega, x) + \hat{\varphi}_{x_2^*}(\omega, x)$ and $\hat{\varphi}_{\lambda x^*}(\omega, x) = \lambda \hat{\varphi}_{x^*}(\omega, x)$ for all $\omega \in \Omega$. Thus $x^* \to \hat{\varphi}_{x^*}(\omega, x)$ is a sublinear function. In addition it is bounded and so it is continuous. Let

$$F(\omega, x) = \{ z \in X \colon (x^*, z) \leqslant \hat{\varphi}_{x^*}(\omega, x) \text{ for all } x^* \in X^* \}.$$

Using the continuity of $\hat{\varphi}_{\cdot}(\omega, x)$ we can write that

$$F(\omega, x) = \bigcap_{n \geqslant 1} \{ z \in X \colon (x_n^*, Z) \leqslant \hat{\varphi}_{x_n^*}(\omega, x) \},$$

where $\{x_n^*\}_{n \geqslant 1}$ is dense in X^*. So $(\omega, x) \to F(\omega, x)$ is measurable with values in $P_{fc}(X)$. Then we have

$$\sigma_{T(f)}(x^*) = \int_\Omega \varphi_{x^*}(\omega, f(\omega))\, d\mu(\omega) = \int_\Omega \sigma_{F(\omega, f(\omega))}(x^*)\, d\mu(\omega)$$

$$= \sigma_{\int_\Omega F(\omega, f(\omega))}(x^*) \Rightarrow T(f) = \text{cl} \int_\Omega F(\omega, f(\omega))\, d\mu(\omega).$$

Also note that

$$\int_\Omega F(\omega, f(\omega)) \, d\mu(\omega) \subseteq \int_\Omega \|f(\omega)\| W(\omega) \, d\mu(\omega) \Rightarrow \int_\Omega \sigma_{F(\omega, f(\omega))}(x^*) \, d\mu(\omega)$$

$$\leqslant \int_\Omega \|f(\omega)\| \sigma_{W(\omega)}(x^*) \, d\mu(\omega).$$

Since this is true for all $f \in L_X^1(\Omega)$, from Lemma 4.2 of Hiai [17] we deduce that $\sigma_{F(\omega, f(\omega))}(x^*) \leqslant \|f(\omega)\| \sigma_{W(\omega)}(x^*)$ for all $\omega \in \Omega \setminus N_{x^*}$ with $\mu(N_{x^*}) = 0$. For the dense set $\{x_n^*\}_{n \geqslant 1}$ clearly we can say that $\sigma_{F(\omega, f(\omega))}(x_n^*) \leqslant \|f(\omega)\| \sigma_{W(\omega)}(x_n^*)$ μ-a.e. for all $n \geqslant 1$. Then from the continuity of the support functions we deduce that $\sigma_{F(\omega, f(\omega))}(x^*) \leqslant \|f(\omega)\| \sigma_{W(\omega)}(x^*)$ μ-a.e. for all $x^* \in X^*$. Since both sets are closed, convex we conclude that $F(\omega, f(\omega)) \subseteq \|f(\omega)\| W(\omega)$ μ-a.e. So $F(\omega, f(\omega))$ has values in $P_{wkc}(X)$ for μ-almost all $\omega \in \Omega$. Hence using the lemma and Theorem 4.2 of this paper we get $\int_\Omega F(\omega, f(\omega)) \, d\mu(\omega) \in P_{wkc}(X)$ and so $T(f) = \int_\Omega F(\omega, f(\omega)) \, d\mu(\omega)$.

Finally we have to show that $F(\omega, \cdot)$ is l.s.c. from X into $(X, w) = X$ with the weak topology, and that $F(\omega, 0) = \{0\}$ μ-a.e. The second conclusion is immediate because we have that

$$\sigma_{F(\omega, 0)}(x^*) = \varphi_{x^*}(\omega, 0) = 0 \quad \mu\text{-a.e.}$$

Since this is true for all $x^* \in X^*$ we deduce that $F(\omega, 0) = \{0\}$ μ-a.e.

Next note that for $x \in B_1 = $ unit ball of X we have that $F(\omega, x) \subseteq W(\omega)$. Also we know that $x \to \sigma_{F(\omega, x)}(x^*)$ is l.s.c. for all $\omega \in \Omega$ and all $x^* \in X^*$. So we can invoke Theorem II-21 of Castaing and Valadier [6] and deduce that, outside a μ-null set N, $F(\omega, \cdot)$ is l.s.c. from B_1 into $(X, w) = $ the space X with the weak topology. By modifying $F(\cdot, x)$ appropriately on N (for example, set $F(\omega, x) = W(\omega)$ for all $\omega \in N$) we can say that $F(\omega, \cdot)$ is l.s.c. from B_1 to (X, w) for all $\omega \in \Omega$. We will show that $F(\omega, \cdot)$ is l.s.c. on all of X. Note that $F(\omega, \cdot)$ is positively homogeneous since $T(\cdot)$ is. So if $x_n \xrightarrow{s} x$ and recalling that the weak topology on $W(\omega)$ is metrizable we have that

$$\text{w-}\lim_{n \to \infty} F(\omega, x_n) = \|x\| \frac{1}{\|x\|} \text{w-}\lim_{n \to \infty} F(\omega, x_n) = \|x\| \text{w-}\lim_{n \to \infty} \frac{1}{\|x\|} F(\omega, x_n)$$

$$= \|x\| \text{w-}\lim_{n \to \infty} F\left(\omega, \frac{x_n}{\|x_n\|}\right).$$

Since $x_n / \|x_n\| \to x / \|x\|$ and $F(\omega, \cdot)$ is l.s.c. from B_1 to X_w we have

$$\text{w-}\lim_{n \to \infty} F\left(\omega, \frac{x_n}{\|x_n\|}\right) \supseteq F\left(\omega, \frac{x}{\|x\|}\right) = \frac{1}{\|x\|} F(\omega, x).$$

Thus we get that

$$\text{w-}\lim_{n \to \infty} F(\omega, x_n) \supseteq \|x\| \frac{1}{\|x\|} F(\omega, x) = F(\omega, x)$$

which by Delahaye and Denel [10] tells us that $F(\omega, \cdot)$ is l.s.c. from X into X_w for all $\omega \in \Omega$. Q.E.D.

Next we will prove the announced multivalued generalization of the Dunford-Pettis theorem. Following Costé and Pallu de la Barrière [8] we will denote by $H(X, X^*)$ the closed, convex, weakly locally compact, nonempty subsets of X, which contain no straight line. Our result also generalizes significantly Theorem 1 of Castaing [4].

So we have

THEOREM 3.2. *If* $T: L^1(\Omega) \to P_{fc}(X)$ *is linear (i.e. additive and homogeneous),* $T(f) \subset \|f\|_1 Q$, *where* Q *is in* $H(X, X^*)$ *and is also bounded, and* $f \to \sigma_{T(f)}(x^*)$ *is continuous for all* $x^* \in X^*$, *then there exists* $F: \Omega \to H(X, X^*)$ *scalarly integrable s.t.* $T(f) = \mathrm{cl} \int_\Omega f(\omega) F(\omega) \, d\mu(\omega)$ *for all* $f(\cdot) \in L^1(\Omega)$.

PROOF. Consider the map $A \to T(\chi_A)$. We claim that this is a weak set valued measure in the sense of Costé and Pallu de la Barrière [8]. For that purpose let $\{A_n\}_{n \geq 1}$ be a sequence of mutually disjoint Σ-sets, and let $A = \bigcup_{n=1}^\infty A_n$. Let $S_N = \bigcup_{n=1}^N A_n$. Then $T(\chi_A) = T(\chi_{S_N} + \chi_{S_N^c}) = T(\chi_{S_N}) + T(\chi_{S_N^c})$. Note that $T(\chi_{S_N}) = \sum_{n=1}^N T(\chi_{A_n})$. So we have

$$T(\chi_A) = \sum_{n=1}^N T(\chi_{A_n}) + T(\chi_{S_N^c}) \Rightarrow \sigma_{T(\chi_A)} = \sum_{n=1}^N \sigma_{T(\chi_{A_n})} + \sigma_{T(\chi_{S_N^c})}.$$

But $\chi_{S_N^c} \overset{L^1(\Omega)}{\to} 0$ as $N \to \infty$ and so $\sigma_{T(\chi_{S_N^c})}(x^*) \to 0$ as $N \to \infty$ for all $x^* \in X^*$. Also $\chi_{S_N} \overset{L^1(\Omega)}{\to} \chi_A$. So for all $x^* \in X^*$

$$\sigma_{T(\chi_{S_N})}(x^*) = \sum_{n=1}^N \sigma_{T(\chi_{A_n})}(x^*) \to \sigma_{T(\chi_A)}(x^*) \quad \text{as } N \to \infty.$$

Therefore

$$\sigma_{T(\chi_A)}(x^*) = \lim_{N \to \infty} \sum_{n=1}^N \sigma_{T(\chi_{A_n})}(x^*) = \sum_{n=1}^\infty \sigma_{T(\chi_{A_n})}(x^*).$$

Furthermore $T(\chi_\cdot)$ is μ-continuous, because if $\mu(A) = 0$ we have that $T(\chi_A) \subseteq \|\chi_A\|_1 Q = 0 . Q = \{0\}$. So we can apply Theorem 3.1 of Costé and Pallu de la Barrière [8] and get that there exists $F: \Omega \to H(X, X^*)$ scalarly measurable (and so measurable by Theorem III-37 of Castaing and Valadier [6]) and scalarly quasintegrable and for which we have

$$\sigma_{T(\chi_A)}(x^*) = \int_A \sigma_{F(\omega)}(x^*) \, d\mu(\omega) = \sigma_{\int_\Omega \chi_A(\omega) F(\omega) \, d\mu(\omega)}(x^*)$$

for all $x^* \in X^*$. In fact, $\sigma_{F(\cdot)}(x^*)$ is integrable since Q is bounded, i.e. $F(\cdot)$ is scalarly integrable. So we have

$$T(\chi_A) = \mathrm{cl} \int_\Omega \chi_A(\omega) F(\omega) \, d\mu(\omega).$$

Now let $s(\cdot) = \sum_{k=1}^N \chi_{A_k}(\cdot) r_k$ be a nonnegative simple function. Then

$$T(s) = \sum_{k=1}^N T(\chi_{A_k} r_k) = \sum_{k=1}^N \mathrm{cl} \int_{A_k} r_k F(\omega) \, d\mu(\omega)$$

$$= \mathrm{cl} \sum_{k=1}^N \int_{A_k} r_k F(\omega) \, d\mu(\omega) = \mathrm{cl} \int_\Omega s_n(\omega) F(\omega) \, d\mu(\omega).$$

For an arbitrary simple function $s(\cdot)$ write $s = s^+ - s^-$. Because of the additivity of $T(\cdot)$ we have that

$$T(s) = T(s^+ - s^-) = T(s^+) - T(s^-)$$

$$= \operatorname{cl} \int_\Omega s^+(\omega) F(\omega) \, d\mu(\omega) - \operatorname{cl} \int_\Omega s^-(\omega) F(\omega) \, d\mu(\omega)$$

$$= \operatorname{cl}\left[\int_\Omega s^+(\omega) F(\omega) \, d\mu(\omega) - \int_\Omega s^-(\omega) F(\omega) \, d\mu(\omega) \right].$$

Next let $f(\cdot) \in L^1(\Omega)$ and let $\{s_n(\cdot)\}_{n \geqslant 1}$ be simple functions s.t.

$$s_n \overset{L^1(\Omega)}{\to} f, \quad s_n^+ \overset{L^1(\Omega)}{\to} f^+ \quad \text{and} \quad s_n^- \overset{L^1(\Omega)}{\to} f^-$$

and also converge pointwise. Then $\sigma_{T(s_n)}(x^*) \to \sigma_{T(f)}(x^*)$ for all $x^* \in X^*$. Note that

$$\sigma_{T(s_n)}(x^*) = \sigma_{T(s_n^+ - s_n^-)}(x^*) = \sigma_{T(s_n^+) - T(s_n^-)}(x^*)$$

$$= \sigma_{T(s_n^+)}(x^*) + \sigma_{-T(s_n^-)}(x^*) = \sigma_{T(s_n^+)}(x^*) + \sigma_{T(s_n^-)}(-x^*).$$

But we already saw that

$$\sigma_{T(s_n^+)}(x^*) = \sigma_{\operatorname{cl} \int_\Omega s_n^+ F}(x^*) \quad \text{and} \quad \sigma_{T(s_n^-)}(x^*) = \sigma_{\operatorname{cl} \int_\Omega s_n^- F}(-x^*).$$

At this point observe that

$$\left| \sigma_{\operatorname{cl} \int_\Omega s_n^+ F}(x^*) - \sigma_{\operatorname{cl} \int_\Omega f^+ F}(x^*) \right| = \left| \int_\Omega \left(\sigma_{s_n^+(\omega) F(\omega)}(x^*) - \sigma_{f^+(\omega) F(\omega)}(x^*) \right) d\mu(\omega) \right|$$

$$= \left| \int_\Omega \left(s_n^+(\omega) \sigma_{F(\omega)}(x^*) - f^+(\omega) \sigma_{F(\omega)}(x^*) \right) d\mu(\omega) \right|$$

$$\leqslant \int_\Omega |s_n^+(\omega) - f^+(\omega)| \, |\sigma_{F(\omega)}(x^*)| \, d\mu(\omega).$$

Invoking the Lebesgue dominated convergence theorem we get that

$$\lim_{n \to \infty} \left| \int_\Omega \left(s_n^+(\omega) \sigma_{F(\omega)}(x^*) - f^+(\omega) \sigma_{F(\omega)}(x^*) \right) d\mu(\omega) \right|$$

$$= 0 \Rightarrow \sigma_{\operatorname{cl} \int_\Omega s_n^+ F}(x^*) \to \sigma_{\operatorname{cl} \int_\Omega f^+ F}(x^*) \quad \text{as } n \to \infty.$$

Similarly we can get that

$$\sigma_{\operatorname{cl} \int_\Omega s_n^- F}(x^*) \to \sigma_{\operatorname{cl} \int_\Omega f^- F}(x^*) \quad \text{as } n \to \infty$$

for all $x^* \in X^*$. So

$$\sigma_{T(s_n)}(x^*) = \sigma_{T(s_n^+)}(x^*) + \sigma_{T(s_n^-)}(-x^*)$$

$$= \sigma_{\operatorname{cl} \int_\Omega s_n^+ F}(x^*) + \sigma_{\operatorname{cl} \int_\Omega s_n^- F}(-x^*) \to \sigma_{\operatorname{cl} \int_\Omega f^+ F}(x^*) + \sigma_{\operatorname{cl} \int_\Omega f^- F}(-x^*)$$

$$= \sigma_{\operatorname{cl} \int_\Omega f^+ F}(x^*) + \sigma_{-\operatorname{cl} \int_\Omega f^- F}(x^*) = \int_\Omega \sigma_{f^+(\omega) F(\omega) - f^-(\omega) F(\omega)}(x^*) \, d\mu(\omega)$$

$$= \int_\Omega \sigma_{f(\omega) F(\omega)}(x^*) \, d\mu(\omega) = \sigma_{\operatorname{cl} \int_\Omega f F}(x^*).$$

On the other hand, we know that $\sigma_{T(s_n)}(x^*) \to \sigma_{T(f)}(x^*)$ as $n \to \infty$. Thus $\sigma_{T(f)}(x^*) = \sigma_{\operatorname{cl} \int_\Omega f F}(x^*)$ for all $x^* \in X^* \Rightarrow T(f) = \operatorname{cl} \int_\Omega f(\omega) F(\omega) \, d\mu(\omega)$.   Q.E.D.

REMARK. If $Q \in P_{wkc}(X)$, then

$$\int_A F(\omega) \, d\mu(\omega) \subseteq \mu(A)Q = \int_A Q \, d\mu(\omega) \Rightarrow F(\omega) \subseteq Q \quad \mu\text{-a.e.}$$

for all $A \in \Sigma$. So $F(\cdot)$ can be taken to be $P_{wkc}(X)$ valued and integrably bounded and then by Theorem 4.2 of this paper $\int_\Omega f(\omega)F(\omega) \, d\mu(\omega) \in P_{wkc}(X)$ for all $f(\cdot) \in L^1(\Omega)$ and so the representation formula in the above theorem takes the form $T(f) = \int_\Omega f(\omega)F(\omega) \, d\mu(\omega)$.

Next we are going to use Theorem 3.2 to obtain a corresponding representation result for set valued operators acting on $L^\infty(\Omega)$.

THEOREM 3.3. *Assume that X^* is separable too. If $T: L^\infty(\Omega) \to P_{fc}(X)$ is s.t. there exists $\Omega_\varepsilon \subseteq \Omega$ for every $\varepsilon > 0$ with $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ for which $T(f) \subseteq \|f\|_1 Q_\varepsilon$, where $Q_\varepsilon \in P_{wkc}(X)$, and whenever $\operatorname{supp} f \subseteq \Omega_\varepsilon$ and $f \to \sigma_{T(f)}(x^*)$ is continuous on $L^\infty(\Omega_\varepsilon)$ for all $x^* \in X^*$ and all $\varepsilon > 0$, and for all $f_n \overset{w\text{-}L^1}{\to} f$ we have $\operatorname{w-}\underline{\lim}_{n \to \infty} T(f_n) = T(f)$, then there exists $F: \Omega \to P_{wkc}(X)$ scalarly integrable s.t.*

$$T(f) = \operatorname{cl} \int_\Omega f(\omega)F(\omega) \, d\mu(\omega) \quad \text{for all } f \in L^\infty(\Omega).$$

PROOF. Let $\{\Omega_n\}_{n \geq 1} \subseteq \Sigma$ with $\mu(\Omega_n) \uparrow \mu(\Omega)$ and $\{Q_n\}_{n \geq 1} \subseteq H(X, X^*)$ be sequences of sets in Ω and X respectively, postulated by the hypotheses of the theorem. Then from Theorem 3.2 we know that for all $n \geq 1$ there exist $F_n: \Omega \to P_{wkc}(X)$ scalarly integrable s.t. $T(f) = \operatorname{cl} \int_\Omega f(\omega)F_n(\omega) \, d\mu(\omega)$ for all $f(\cdot) \in L^\infty(\Omega_n)$. Suppose $\mu(\Omega_n \cap \Omega_m) > 0$. For all $B \in [\Sigma \cap (\Omega_n \cap \Omega_m)]$ we have that $\operatorname{cl} \int_B F_n(\omega) \, d\mu(\omega)$ $= \operatorname{cl} \int_B F_m(\omega) \, d\mu(\omega) = T(\chi_B)$. Then Lemma 4.2 of Hiai and Umegaki [18] tells us that

$$S^1_{F_n|\Omega_n \cap \Omega_m} = S^1_{F_n|\Omega_n \cap \Omega_m} \Rightarrow F_n(\omega) = F_m(\omega) \quad \mu_{\Omega_n \cap \Omega_m}\text{-a.e.}$$

To take care of those exceptional μ-null sets and of the case where $\mu(\Omega_n \cap \Omega_m) = 0$ and $F_n(\omega) \neq F_m(\omega)$ there, call those sets N_{nm} and let $N_n = \bigcup_{m=1}^\infty N_{nm}$ and $N' = \bigcup_{n=1}^\infty N_n$. Clearly $\mu(N') = 0$. Also let $N'' = \Omega \setminus \bigcup_{n=1}^\infty \Omega_n$. Again $\mu(N'') = 0$. Set $N = N' \cup N''$ and define

$$F(\omega) = \begin{cases} F_n(\omega) & \text{for } \omega \in \Omega_n \setminus N_n, \\ \{0\} & \text{for } \omega \in N. \end{cases}$$

What preceded the definition of $F(\cdot)$ shows that this is a well-defined multifunction. Our claim is that for all $f \in L^\infty(\Omega)$

$$T(f) = \operatorname{cl} \int_\Omega f(\omega)F(\omega) \, d\mu(\omega).$$

Let $g \in S^1_F$ and consider $\{g_n = \chi_{\Omega_n} g\}_{n \geq 1}$ and $\{f_n = \chi_{\Omega_n} f\}_{n \geq 1}$. Then $f_n \cdot g_n \in L^1_X(\Omega)$ and $T(f_n) = \int_\Omega f_n(\omega)g_n(\omega) \, d\mu(\omega)$. Since $f_n \cdot g_n \overset{L^1_X}{\to} f \cdot g$ we have that $\int_\Omega f_n(\omega)g_n(\omega) \, d\mu(\omega) \overset{s}{\to} \int_\Omega f(\omega)g(\omega) \, d\mu(\omega)$. Also recall that $\operatorname{w-}\underline{\lim}_{n \to \infty} T(f_n) = T(f)$. Hence

$$\int_\Omega f(\omega)g(\omega) \, d\mu(\omega) \in T(f) \Rightarrow \operatorname{cl} \int_\Omega f(\omega)F(\omega) \, d\mu(\omega) \subseteq T(f).$$

Suppose that in fact strict inclusion holds. So there exists $u \in T(f)$ s.t. $u \notin$ cl $\int_\Omega f(\omega) F(\omega) \, d\mu(\omega)$. Then from the strong separation theorem we know that there exist $\varepsilon > 0$ and nonzero $x^* \in X^*$ s.t.

$$\sigma_{\int_\Omega f F}(x^*) = \int_\Omega \sigma_{f(\omega)F(\omega)}(x^*) \, d\mu(\omega) \leqslant (x^*, u) - \varepsilon.$$

Apply Aumann's selection theorem to find $g \in S_F^1$ s.t.

$$\int_\Omega \sigma_{f(\omega)F(\omega)}(x^*) \, d\mu(\omega) = \int_\Omega (x^*, f(\omega)g(\omega)) \, d\mu(\omega)$$

$$= \left(x^*, \int_\Omega f(\omega)g(\omega) \, d\mu(\omega)\right).$$

Thus we have that $(x^*, \int_\Omega f(\omega)g(\omega) \, d\mu(\omega)) \leqslant (x^*, u) - \varepsilon$. Recall that $\int_\Omega f_n(\omega)g_n(\omega) \, d\mu(\omega) \xrightarrow{s} \int_\Omega f(\omega)g(\omega) \, d\mu(\omega)$. So for $n \geqslant n_1$ we have

$$(1) \qquad \left(x^*, \int_\Omega f_n(\omega)g_n(\omega) \, d\mu(\omega)\right) \leqslant \left(x^*, \int_\Omega f(\omega)g(\omega) \, d\mu(\omega)\right) + \varepsilon/3.$$

Once again we recall that w-$\underline{\lim}_n T(f_n) = T(f)$. Hence we can find $u_n \in T(f_n)$ s.t. $u_n \xrightarrow{w} u$. Then for $n \geqslant n_2$ we will have

$$(2) \qquad (x^*, u) - \varepsilon/3 \leqslant (x^*, u_n).$$

Combining (1) and (2) above we deduce that for $n \geqslant \max(n_1, n_2)$

$$(3) \qquad \left(x^*, \int_\Omega f_n(\omega)g_n(\omega) \, d\mu(\omega)\right) \leqslant (x^*, u_n) - \varepsilon/3.$$

Note that $g_n \in S_{F_n}^1$. We claim that $(x^*, f_n(\omega)g_n(\omega)) = \sigma_{f_n(\omega)F_n(\omega)}(x^*)$ μ_{Ω_n}-a.e. Suppose not. Then for some $A \in \Sigma \cap \Omega_n$ with $\mu_{\Omega_n}(A) = \mu(A) > 0$ we have that

$$(4)$$
$$(x^*, f_n(\omega)g_n(\omega)) < \sigma_{f_n(\omega)F_n(\omega)}(x^*) \quad \text{for } \omega \in A$$

$$\Rightarrow \int_{\Omega_n} (x^*, f_n(\omega)g_n(\omega)) \, d\mu(\omega) < \int_{\Omega_n} \sigma_{f_n(\omega)F_n(\omega)}(x^*) \, d\mu(\omega).$$

But $\sigma_{f(\omega)F(\omega)}(x^*) = (x^*, f(\omega)g(\omega))$ μ-a.e. So

$$(5) \qquad \int_{\Omega_n} \sigma_{f(\omega)F(\omega)}(x^*) \, d\mu(\omega) = \left(x^*, \int_{\Omega_n} f(\omega)g(\omega) \, d\mu(\omega)\right)$$

$$\Rightarrow \int_{\Omega_n} \sigma_{f_n(\omega)F_n(\omega)}(x^*) \, d\mu(\omega)$$

$$= \left(x^*, \int_{\Omega_n} f_n(\omega)g_n(\omega) \, d\mu(\omega)\right).$$

From (4) and (5) we get a contradiction. Hence indeed

$$\sigma_{f_n(\omega)F_n(\omega)}(x^*) = (x^*, f_n(\omega)g_n(\omega)) \quad \mu_{\Omega_n}\text{-a.e.}$$

Using that in (3) we get

$$\int_{\Omega_n} \sigma_{f_n(\omega)F_n(\omega)}(x^*)\, d\mu(\omega) \leqslant (x^*, u_n) - \varepsilon/3$$

$$\Rightarrow \sigma_{\int_{\Omega_n} f_n F_n}(x^*) \leqslant (x^*, u_n) - \varepsilon/3$$

$$\Rightarrow u_n \notin T(f_n) = \int_{\Omega_n} f_n(\omega)F_n(\omega)\, d\mu(\omega)$$

for $n \geqslant \max(n_1, n_2)$, which is a contradiction. So finally we conclude that $T(f) = \int_\Omega f(\omega)F(\omega)\, d\mu(\omega)$.   Q.E.D.

We will close this section with another representation theorem for set valued operators on $L^\infty(\Omega)$ which are continuous with respect to the Kuratowski-Mosco convergence of sets, when the arguments in $L^\infty(\Omega)$ converge strongly.

THEOREM 3.4. *Assume that (Ω, Σ, μ) is nonatomic and X is a separable reflexive Banach space. If $T: L^\infty \to P_{fc}(X)$ is linear, $f \to T(f)$ is continuous on $L^\infty(\Omega)$ for the K-M convergence and $T(\chi_A) \subseteq \mu(A)Q$ for all $A \in \Sigma$, where Q is closed and bounded, then there exists $F: \Omega \to P_{fc}(X)$ integrably bounded s.t. $T(f) = \int_\Omega f(\omega)F(\omega)\, d\mu(\omega)$ for all $f \in L^\infty(\Omega)$.*

PROOF. As before we can have that $A \to \sigma_{T(\chi_A)}(x^*)$ is a signed measure for all $x^* \in X^*$. Since $T(\cdot)$ has values in $P_{wkc}(X)$, from Theorem 1.1 of Costé [7] we deduce that $M: A \to T(\chi_A)$ is a strong multimeasure in the sense of that paper.

Also from our boundedness assumption we can see that $M(\cdot)$ is μ-continuous and of bounded variation. Invoking Theorem 4.5 of Hiai [16] we get that $M(\cdot)$ has a set valued Radon-Nikodým derivative $F: \Omega \to P_f(X)$ which is integrably bounded. Because (Ω, Σ, μ) is nonatomic we can take $F(\cdot)$ to be convex valued. Also note that $F(\omega) \subseteq Q$ μ-a.e. So we have

$$T(\chi_A) = \int_A F(\omega)\, d\mu(\omega)$$

with $F: \Omega \to P_{wkc}(X)$ is integrably bounded.

Let

$$s(\cdot) = \sum_{k=1}^N \chi_{A_k}(\cdot)r_k \Rightarrow T(s) = \sum_{k=1}^N T(\chi_{A_k}r_k) = \sum_{k=1}^N r_k T(\chi_{A_k})$$

$$= \sum_{k=1}^N r_k \int_{A_k} F(\omega)\, d\mu(\omega) = \sum_{k=1}^N \int_\Omega \chi_{A_k}(\omega)r_k F(\omega)\, d\mu(\omega)$$

$$= \int_\Omega \sum_{k=1}^N \chi_{A_k}(\omega)r_k F(\omega)\, d\mu(\omega) = \int_\Omega s(\omega)F(\omega)\, d\mu(\omega).$$

Next let $\{s_n\}_{n \geqslant 1}$ be simple functions s.t. $s_n \overset{L^\infty}{\to} f$ (see Hewitt and Stromberg [15, Theorem 11.35, p. 159]). We claim that

$$\int_\Omega s_n(\omega)F(\omega)\, d\mu(\omega) \overset{K-M}{\to} \int_\Omega f(\omega)F(\omega)\, d\mu(\omega).$$

So let $x \in$ w-$\overline{\lim} \int_\Omega s_n(\omega) F(\omega) \, d\mu(\omega)$. Then $x =$ w-$\lim_{k \to \infty} x_{n_k}$ with $x_{n_k} \in \int_\Omega s_{n_k}(\omega) F(\omega) \, d\mu(\omega)$. Let $x_{n_k} = \int_\Omega s_{n_k}(\omega) g_{n_k}(\omega) \, d\mu(\omega)$, where $g_{n_k}(\cdot) \in S_F^1$. Theorem 4.2 of this paper (see also Proposition 3.1 of [24]) tells us that S_F^1 is w-compact in $L_X^1(\Omega)$. So from the Eberlein-Smulian theorem we deduce that there is a subsequence of $\{g_{n_k}(\cdot)\}_{k \geq 1}$ (which for simplicity we denote with the same index) s.t.

$$g_{n_k} \xrightarrow{\text{w-}L_X^1} g \in S_F^1.$$

Then for every $x^* \in X^*$ we have that

$$\left| \left(x^*, \int_\Omega s_{n_k}(\omega) g_{n_k}(\omega) \, d\mu(\omega) \right) - \left(x^*, \int_\Omega f(\omega) g(\omega) \, d\mu(\omega) \right) \right|$$

$$= \left| \left(x^*, \int_\Omega s_{n_k}(\omega) g_{n_k}(\omega) \, d\mu(\omega) \right) - \left(x^*, \int_\Omega f(\omega) g_{n_k}(\omega) \, d\mu(\omega) \right) \right.$$

$$\left. + \left(x^*, \int_\Omega f(\omega) g_{n_k}(\omega) \, d\mu(\omega) \right) - \left(x^*, \int_\Omega f(\omega) g(\omega) \, d\mu(\omega) \right) \right|$$

$$\leq \left| \left(x^*, \int_\Omega (s_{n_k}(\omega) - f(\omega)) g_{n_k}(\omega) \, d\mu(\omega) \right) \right|$$

$$+ \left| \left(x^*, \int_\Omega f(\omega)(g_{n_k}(\omega) - g(\omega)) \, d\mu(\omega) \right) \right|$$

$$= \left| \int_\Omega (s_{n_k}(\omega) - f(\omega))(x^*, g_{n_k}(\omega)) \, d\mu(\omega) \right|$$

$$+ \left| \int_\Omega f(\omega)(x^*, g_{n_k}(\omega) - g(\omega)) \, d\mu(\omega) \right|$$

$$= \left| \langle (s_{n_k} - f) x^*, g_{n_k} \rangle \right| + \left| \langle f x^*, g_{n_k} - g \rangle \right|,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets between $L_{X^*}^\infty(\Omega)$ and $L_X^1(\Omega)$. Note that

$$s_{n_k} x^* \xrightarrow{\text{s-}L_{X^*}^\infty} f x^* \quad \text{and} \quad g_{n_k} \xrightarrow{\text{w-}L_X^1} g \quad \text{as } k \to \infty.$$

So passing to the limit we get that

$$\left(x^*, \int_\Omega s_{n_k}(\omega) g_{n_k}(\omega) \, d\mu(\omega) \right) \to \left(x^*, \int_\Omega f(\omega) g(\omega) \, d\mu(\omega) \right)$$

$$\Rightarrow x_{n_k} = \int_\Omega s_{n_k}(\omega) g_{n_k}(\omega) \, d\mu(\omega)$$

$$\xrightarrow{\text{w}} \int_\Omega f(\omega) g(\omega) \, d\mu(\omega) \in \int_\Omega f(\omega) F(\omega) \, d\mu(\omega).$$

On the other hand, we also have that $x_{n_k} \xrightarrow{\text{w}} x$. So $x = \int_\Omega f(\omega) g(\omega) \, d\mu(\omega)$ which implies that

$$(6) \qquad \text{w-}\lim_{n \to \infty} \int_\Omega s_n(\omega) F(\omega) \, d\mu(\omega) \subseteq \int_\Omega f(\omega) F(\omega) \, d\mu(\omega).$$

Next let $x \in \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega)$. Then there exists $g(\cdot) \in S_F^1$ s.t. $x = \int_{\Omega} f(\omega) g(\omega) \, d\mu(\omega)$. Let

$$x_n = \int_{\Omega} s_n(\omega) g(\omega) \, d\mu(\omega) \in \int_{\Omega} s_n(\omega) F(\omega) \, d\mu(\omega).$$

We have

$$\|x_n - x\| \leqslant \int_{\Omega} |s_n(\omega) - f(\omega)| \|g(\omega)\| \to 0 \quad \text{as } n \to \infty.$$

So $x \in$ s-$\underline{\lim}_{n \to \infty} \int_{\Omega} s_n(\omega) F(\omega) \, d\mu(\omega)$ which means that

(7) $\int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega) \subseteq$ s-$\underline{\lim}_{n \to \infty} \int_{\Omega} s_n(\omega) F(\omega) \, d\mu(\omega).$

From (6) and (7) we get that

$$\int_{\Omega} s_n(\omega) F(\omega) \, d\mu(\omega) \overset{\text{K-M}}{\to} \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega)$$

$$\Rightarrow T(s_n) \overset{\text{K-M}}{\to} \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega).$$

On the other hand, since $T(\cdot)$ is continuous for the K-M convergence on $L^\infty(\Omega)$ we have that $T(s_n) \overset{\text{K-M}}{\to} T(f)$. Recall that the Kuratowski-Mosco limit is unique. So $T(f) = \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega)$.   Q.E.D.

4. Integrable multifunctions. In this section we will examine some properties of the representation integrals obtained in the previous section and then we will prove a differentiability kind of result for a large class of multifunctions and we will determine a necessary and sufficient condition for S_F^1 to be w-compact in L_X^1, generalizing an earlier result by Diestel [11].

In the first result we characterize the extremal points of a representable set valued operator $T: L^1 \to P_{fc}(X)$.

PROPOSITION 4.1. *If* $T(f) = \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega)$, *where* $F: \Omega \to P_{wkc}(X)$ *is integrably bounded, then* $\text{ext } T(f) \subseteq \int_{\Omega} f(\omega) \text{ext } F(\omega) \, d\mu(\omega)$.

PROOF. Let $x \in \text{ext } T(f)$. Then $x = \int_{\Omega} f(\omega) g(\omega) \, d\mu(\omega)$ for $g(\cdot) \in S_F^1$. Since we know that $S_{\text{ext } F}^1 = \text{ext } S_F^1$ from Castaing and Valadier [6], if $g \notin S_{\text{ext } F}^1$ then there must be $g_1, g_2 \in S_F^1$ s.t. $g = \frac{1}{2} g_1 + \frac{1}{2} g_2$. Hence

$$x = \int_{\Omega} f(\omega) g(\omega) \, d\mu(\omega) = \int_{\Omega} f(\omega) \left[\frac{1}{2} g_1(\omega) + \frac{1}{2} g_2(\omega) \right] d\mu(\omega)$$

$$= \frac{1}{2} \int_{\Omega} f(\omega) g_1(\omega) \, d\mu(\omega) + \frac{1}{2} \int_{\Omega} f(\omega) g_2(\omega) \, d\mu(\omega) = \frac{x_1 + x_2}{2}$$

with $x_1, x_2 \in T(f) = \int_{\Omega} f(\omega) F(\omega) \, d\mu(\omega)$. But this then contradicts the extremality of x.   Q.E.D.

We have another characterization of the exposed points of a set valued operator $T(\cdot)$ acting on $L^1(\Omega)$.

Assume that X has the R.N.P.

PROPOSITION 4.2. *If* $T: L^1 \to P_{fc}(X)$ *is additive,* $f \to \sigma_{T(f)}(x^*)$ *is continuous for all* $x^* \in X^*$, $T(f_0) \in P_{wkc}(X)$ *for* $f_0 \in L^1(\Omega)$ *and* $|T(\chi_B f_0)| \leqslant \mu(B)\|f_0\|_1$, *where* $B \in \Sigma$, *and* $x \in \exp T(f_0)$, *then there exists* $g_{f_0}(\cdot) \in L^1_X(\Omega)$ *s.t.* $x = \int_\Omega g_{f_0}(\omega) \, d\mu(\omega)$.

REMARK. Note that such a set valued operator is not necessarily representable and so we cannot get the proposition as a consequence of one of the representation theorems of §3.

PROOF. As before we can show that $A \to \sigma_{T(\chi_A f_0)}(x^*)$ for all $x^* \in X^*$ is a signed measure. Since $T(f_0) \in P_{wkc}(X)$, Theorem 1.1 of Costé [7] tells us that M: $A \to T(\chi_A f_0)$ is a strong multimeasure which is clearly μ-continuous and of bounded variation. So we can use Proposition 2.1 of Hiai [16] and get a vector measure m_{f_0}: $\Sigma \to X$ s.t. $m_{f_0}(\Omega) = x$. Note that $m_{f_0}(\cdot)$ being a measure selector of $M(\cdot)$ is μ-continuous and of bounded variation. Since X has the R.N.P. we deduce that there exists $g_{f_0}(\cdot) \in L^1_X(\Omega)$ s.t. $x = \int_\Omega g_{f_0}(\omega) \, d\mu(\omega)$.   Q.E.D.

Now we will have the differentiation result for multifunctions which extends the single valued result of Gelfand (see [21]) and the earlier multivalued results of Artstein [2], Castaing [5] and Hermes [14]. Assume X is reflexive.

THEOREM 4.1. *If* $\Phi: [0, T] \to P_{fc}(X)$ *is measurable and for all* $x^* \in X^*$ *and all* t_0, $t_1 \in [0, T]$ *we have that*

$$\left|\sigma_{\Phi(t_0)}(x^*) - \sigma_{\Phi(t_1)}(x^*)\right| \leqslant \|x^*\||t_1 - t_0|,$$

then there exists $F: \Omega \to P_{fc}(X)$ *scalarly integrable s.t.* $\Phi(t_1) = \Phi(t_0) + \mathrm{cl} \int_{t_0}^{t_1} F(t) \, dt$ *for all* $t_0, t_1 \in [0, T]$.

PROOF. From Dinculeanu [13] we know that to $\sigma_{\Phi(t)}(x^*)$ corresponds a measure $m_{x^*}(\cdot)$ s.t.

$$m_{x^*}[t_0, t_1] = \sigma_{\Phi(t_1)}(x^*) - \sigma_{\Phi(t_0)}(x^*)$$

and

$$(*) \qquad |m_{x^*}|(A) \leqslant \|x^*\|\lambda(A), \qquad A \in \Sigma = \text{Lebesgue sets of } [0, T],$$

where $\lambda(\cdot)$ is the Lebesgue measure on $[0, T]$. From the Radon-Nikodým Theorem we know that there exists $\varphi(x^*, \cdot) \in L^1$ s.t.

$$(**) \qquad\qquad m_{x^*}(A) = \int_A \varphi(x^*, t) \, dt$$

for every Lebesgue measurable set A in $[0, T]$. We claim that $\varphi(x^*, \cdot) \in L^\infty$. Suppose not. Then for every $n \geqslant 1$ there exist $A_n \in \Sigma = $ Lebesgue measurable sets in $[0, 1]$ with $\mu(A_n) > 0$ s.t. $|\varphi(x^*, t)| > n$ for $t \in A_n$. Hence we have that

$$|m_{x^*}|(A_n) = \int_{A_n} |\varphi(x^*, t)| dt > n \cdot \lambda(A_n)$$

$$\Rightarrow \frac{|m_{x^*}|(A_n)}{\lambda(A_n)} > n$$

which shows that $\{|m_{x^*}|(A)/\lambda(A): A \in \Sigma, \lambda(A) > 0\}$ is unbounded, contradicting $(*)$. Thus $\varphi(x^*, \cdot) \in L^\infty$. From $(**)$ we see that $x^* \to \varphi(x^*, t)$ is sublinear for $t \in [0, T] \setminus N_{x^*}$, where $\lambda(N_{x^*}) = 0$. By a lifting argument as in the proof of Theorem

3.1 we can get that $x^* \to \hat{\varphi}(x^*, t) = \rho[\varphi(x^*, t)]$ is sublinear for all $t \in [0, T]$. Also it is continuous. So applying Hörmander's theorem we can find $F: \Omega \to P_{fc}(X)$ s.t. $\hat{\varphi}(x^*, t) = \sigma_{F(t)}(x^*)$. So $F(\cdot)$ is scalarly integrable (and so $F(\cdot)$ is measurable by Theorem III-37 of Castaing and Valadier [6]). Hence we have

$$m_{x^*}(A) = \int_A \sigma_{F(t)}(x^*) \, dt$$

$$\Rightarrow \sigma_{\Phi(t_1)}(x^*) = \sigma_{\Phi(t_0)}(x^*) + \int_{t_0}^{t_1} \sigma_{F(t)}(x^*) \, dt$$

$$\Rightarrow \sigma_{\Phi(t_1)}(x^*) = \sigma_{\Phi(t_0)}(x^*) + \sigma_{\int_{t_0}^{t_1} F(t) \, dt}(x^*).$$

Since this is true for all $x^* \in X^*$ we conclude that

$$\Phi(t_1) = \Phi(t_0) + \mathrm{cl} \int_{t_0}^{t_1} F(t) \, dt. \qquad \text{Q.E.D.}$$

We will close the paper with a result that we have already used in several places in this work. This is a necessary and sufficient condition for the set of integrable selectors of a multifunction to be w-compact in $L_X^1(\Omega)$. The sufficiency part was first obtained by the author in [24]. The necessity part is new. Furthermore the sufficiency part generalizes the result of Diestel [11] and provides a different approach to the problem; namely instead of the Davis-Figiel-Johnson-Pełczynski factorization method (see [11]) we use techniques from the general theory of multifunctions and the celebrated James theorem for weak compactness.

Assume that (Ω, Σ, μ) is nonatomic and X^* is separable.

THEOREM 4.2. *If $F: \Omega \to P_f(X)$ is integrably bounded, then S_F^1 is w-compact in $L_X^1(\Omega)$ if and only if $F(\cdot)$ has values in $P_{wkc}(X)$ μ-a.e.*

PROOF. Assume S_F^1 is w-compact in $L_X^1(\Omega)$. Because (Ω, Σ, μ) is nonatomic from Cuong [9] we know that S_F^1 is weakly dense in $\overline{\mathrm{conv}} \, S_F^1 = S_{\overline{\mathrm{conv}} \, F}^1$. Since S_F^1 is w-closed we deduce that $S_F^1 = S_{\overline{\mathrm{conv}} \, F}^1 \Rightarrow F(\omega) = \overline{\mathrm{conv}} \, F(\omega)$ μ-a.e. So $F(\cdot)$ is μ-a.e. convex valued.

Next let $x^* \in X^*$. Then from James' theorem for some $\hat{f} \in S_F^1$ we have that

$$\sigma_{S_F^1}(x^*) = \langle x^*, \hat{f} \rangle = \left(x^*, \int_\Omega \hat{f}(\omega) \, d\mu(\omega) \right).$$

Also from Theorem 2.2 of Hiai and Umegaki [18] we can easily see that

$$\sigma_{S_F^1}(x^*) = \int_\Omega \sigma_{F(\omega)}(x^*) \, d\mu(\omega).$$

So we have

$$\int_\Omega \left[\sigma_{F(\omega)}(x^*) - (x^*, \hat{f}(\omega)) \right] d\mu(\omega) = 0.$$

But note that since $\hat{f}(\cdot) \in S_F^1$ and $F(\cdot)$ is μ-a.e. convex valued we have that $(x^*, \hat{f}(\omega)) \leqslant \sigma_{F(\omega)}(x^*)$ μ-a.e. Hence,

$$\sigma_{F(\omega)}(x^*) = (x^*, \hat{f}(\omega))$$

for all $\omega \in \Omega \setminus N_{x^*}$, where $\mu(N_{x^*}) = 0$. Since X^* is separable and $\sigma_{F(\omega)}(\cdot)$ is continuous we can easily see that

$$\sigma_{F(\omega)}(x^*) = (x^*, \hat{f}(\omega)) \quad \mu\text{-a.e.}$$

with the exceptional μ-null set being independent of $x^* \in X^*$. So invoking James' theorem we conclude that $F(\omega) \in P_{wkc}(X)$ μ-a.e.

Now assume that $F(\omega) \in P_{wkc}(X)$ μ-a.e.

By setting $F(\omega) = \{0\}$ on the exceptional μ-null set we can assume without any loss of generality that $F(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega$. From the Dinculeanu-Foiaş theorem (see Dinculeanu [13] or Ionescu-Tulcea [20]) we know that $[L_X^1(\Omega)]^* = L_{X_w^*}^\infty(\Omega)$. Letting $g \in L_{X_w^*}^\infty(\Omega)$ we have

$$\sigma_{S_F^1}(g) = \sup_{f \in S_F^1} \langle g, f \rangle = \sup_{f \in S_F^1} \int_\Omega (g(\omega), f(\omega)) \, d\mu(\omega).$$

Using Theorem 2.2 of Hiai and Umegaki [18] we get that

$$\sup_{f \in S_F^1} \int_\Omega (g(\omega), f(\omega)) \, d\mu(\omega) = \int_\Omega \sup_{x \in F(\omega)} (g(\omega), x) \, d\mu(\omega).$$

Let

$$R(\omega) = \left\{ \hat{x} \in F(\omega) : \sup_{x \in F(\omega)} (g(\omega), x) = \sigma_{F(\omega)}(g(\omega)) = (g(\omega), \hat{x}) \right\}.$$

Since $F(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega$, we have that $R(\omega) \neq \varnothing$ for all $\omega \in \Omega$. Set $\varphi(\omega, x) = \sigma_{F(\omega)}(g(\omega)) - (g(\omega), \hat{x})$. Clearly $\varphi(\cdot_3 \cdot)$ is a Carathéodory function and so is jointly measurable. Then observe that $\mathrm{Gr}\, R = \{(\omega, \hat{x}) \in \Omega \times X : \varphi(\omega, \hat{x}) = 0\}$, $\mathrm{Gr}\, F \in \Sigma \times B(X)$. So applying Aumann's selection theorem we can find $\hat{f} : \Omega \to X$ measurable s.t. $\hat{f}(\omega) \in R(\omega)$ for all $\omega \in \Omega$. Note that $\hat{f} \in S_F^1$. Hence we have

$$\int_\Omega \sup_{x \in F(\omega)} (g(\omega), x) \, d\mu(\omega) = \int_\Omega (g(\omega), \hat{f}(\omega)) \, d\mu(\omega)$$

$$\Rightarrow \sigma_{S_F^1}(g) = \langle g, \hat{f} \rangle.$$

Since $g \in L_{X_w^*}^\infty(\Omega)$ was arbitrary, we conclude using James' theorem that S_F^1 is w-compact in $L_X^1(\Omega)$.   Q.E.D.

REMARK. The nonatomicity of (Ω, Σ, μ) and the separability of X^* are not needed in the sufficiency part.

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